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Note

Randomly planar graphs

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Abstract

A graph G is *randomly planar* if every planar embedding of every connected subgraph of G can be extended to a planar embedding of G. We classify these graphs.

1. Introduction

Many properties of graphs have been 'randomized' by various mathematicians. Examples include the notions of randomly eulerian [2], randomly traceable [1], randomly matchable [4], and randomly decomposable [3] graphs. We continue in this vein by randomizing planarity. Throughout this note, G will denote a finite connected graph with labelled vertices. For the sake of brevity, we will only provide sketches of proofs.

Definition 1. A graph G is *randomly planar* if for any connected subgraph H of G, every planar embedding of H can be extended to a planar embedding of G.

Intuitively, a graph is randomly planar if a planar embedding of it can be built up in any manner whatsoever. With such a graph, one can add edges to an embedding one at a time, and no matter how the embedding proceeds, it is always possible to finish.

The requirement that the subgraph H be connected is imposed in order to include a larger class of graphs but can be omitted to give a stronger property.

Definition 2. A graph G is *strongly randomly planar* if for any subgraph H of G, every planar embedding of H can be extended to a planar embedding of G.

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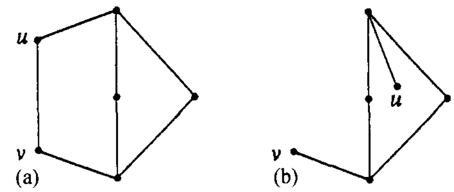

Fig. 1. A planar-forbidden graph and an embedding of one of its subgraphs.

We permit H to be disconnected in this setting. Note that strong random planarity implies random planarity.

We concentrate on embeddings of graphs into the plane and will not discuss the natural generalization to embeddings in other surfaces. In fact, it is not hard to see that a graph is randomly embeddable (resp. strongly randomly embeddable) into a particular surface if and only if it is randomly planar (resp. strongly randomly planar).

Definition 3. A graph is *planar-forbidden* if it is the union of a cycle and a path of length at least 3 between two (possibly equal) vertices of the cycle that is internally disjoint from the cycle (see Fig. 1(a)).

The next lemma makes clear the motivation for this terminology.

Lemma 4. *A graph is randomly planar if and only if it contains no planar-forbidden subgraphs.*

Proof. Fig. 1 demonstrates why a randomly planar graph cannot contain a planar-forbidden subgraph.

Now suppose that G is a graph that is not randomly planar. Then there is some connected subgraph H of G with a planar embedding and an edge uv of G so that u and v are vertices of H and the embedding of H is not extendible to an embedding of $H \cup uv$. It follows that H must contain a cycle C such that u and v lie in distinct path-components of the plane minus the image of C under the planar embedding of H. Now it is possible to find a path in G passing through u and v that starts and ends on C and is internally disjoint from C. The union of this path and C is a planar-forbidden graph. $\square$

Recall that an edge e of a graph G is a bridge if $G - e$ has more components than G. Equivalently, e is a bridge if and only if it lies on no cycles of G. Also, the edge-blocks of G are the connected components of the graph formed by removing all bridges from G, so edge-blocks are maximal connected subgraphs that contain no bridges.

Edge-blocks hold the key to classifying randomly planar graphs.

Theorem 5. *A graph is randomly planar if and only if each of its edge-blocks is randomly planar.*

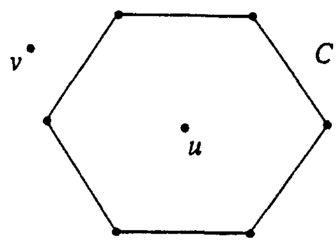

Fig. 2. An embedding of $C \cup u \cup v$.

Proof. Any planar-forbidden subgraph of a graph must lie entirely within some edge-block of that graph. □

Because of Theorem 5, we can focus on randomly planar graphs without any bridges. We consider two cases in the following two lemmas.

Lemma 6. *If G is any randomly planar graph without bridges and having at most two mutually internally disjoint paths between any pair of vertices, then $G \cong K_1$ or $G \cong C_n$ for some $n \geqslant 3$.*

Proof. If G has no cycles then it must be K_1, since this is the only tree without bridges. Now suppose that G has a cycle C but that $G \neq C$. Then there exists some edge uv of G with u in C and uv not in C. Since uv is not a bridge, there is a path P in G that is internally disjoint from C between v and some vertex w of C. Possibly $u = w$, but this implies the existence of a planar-forbidden subgraph. When $u \neq w$, there are three mutually disjoint paths from u to w: $uv \cup P$ and two that lie in C. By contradiction, $G = C$. □

Lemma 7. *If G is a randomly planar graph without bridges and having vertices u and v with at least three mutually internally disjoint paths between them, then $G \cong K_4$, $G \cong K_{2,n}$ with $n \geqslant 3$, or $G \cong K_{1,1,n}$ with $n \geqslant 2$.*

Proof. Let H be a subgraph of G formed by a maximal collection of mutually internally disjoint paths from u to v. If any of these paths had length at least 3, then this path and two others would form a planar-forbidden subgraph of G, contradicting Lemma 4. Hence all of these paths have length at most 2, so $H \cong K_{2,n}$ with $n \geqslant 3$, or $H \cong K_{1,1,n}$ with $n \geqslant 2$.

In the first case, it can be shown that in fact $H = G$. In the second case, it can be shown that either $H = G$ or $G \cong K_4$. □

We are now ready to classify randomly planar graphs.

Theorem 8. *A graph is randomly planar if and only if each of its edge-blocks is isomorphic to K_1, C_n, K_4, $K_{2,n}$ with $n \geqslant 3$, or $K_{1,1,n}$ with $n \geqslant 2$.*

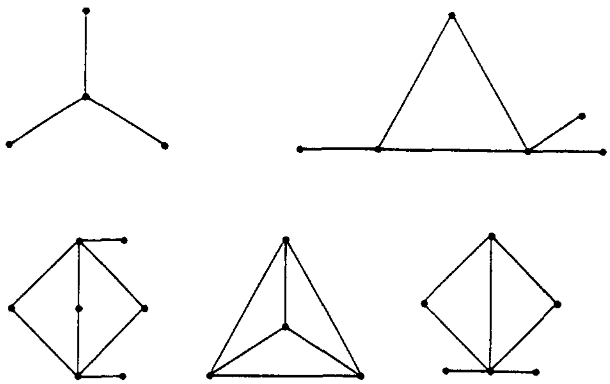

Fig. 3. Strongly randomly planar graphs.

Proof. By inspection, each of the listed graphs is randomly planar, so by Theorem 5, any graph whose edge-blocks consist of the listed graphs must be randomly planar.

Conversely, if G is randomly planar, then every edge-block is randomly planar and has no bridges. Now either Lemma 6 or Lemma 7 applies to each edge-block; in either case, every edge-block is isomorphic to one of the graphs listed. □

We can now easily classify strongly randomly planar graphs.

Lemma 9. *A graph is strongly randomly planar if and only if every edge of G is incident with at least one vertex of every cycle in G.*

Proof. A glance at Fig. 2 shows why a strongly randomly planar graph cannot contain a cycle and an edge that shares no vertex with the cycle. The converse follows from an argument similar to the one given for Lemma 4. □

Theorem 10. *A graph is strongly randomly planar if and only if it is a tree, a cycle with pendent edges, $K_{2,n}$ with pendent edges at the doubleton partite set, $K_{1,1,n}$ with pendent edges at the singleton partite sets, or K_4 (see Fig. 3).*

Proof. It is easy to see by Lemma 9 that the graphs listed are all strongly randomly planar. Conversely, it is straightforward to find all the graphs listed in Theorem 8 that satisfy the property that every cycle and every edge share at least one vertex. □

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